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Pseudoeffect Algebras. II. Group Representations

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This paper is the continuation of the previous paper by Dvurečenskij and Vetterlein (2001), *Int. J. Theor. Phys.* **40**(3). We show that any pseudoeffect algebra fulfilling a certain property of Riesz type is representable by a unit interval of some (not necessarily Abelian) partially ordered group. The relation of pseudoeffect to pseudo-MV algebras is made clear, and the ℓ-group representation theorem for the latter structure is re-proved.

With this paper, we continue the work Dvurečenskij and Vetterlein (2001), where we introduced a new algebraic structure called a pseudoeffect algebra. Section and theorem numbers continue from those of the paper mentioned above.

5. REPRESENTATION OF PSEUDOEFFECT ALGEBRAS BY UNIT INTERVALS OF po-GROUPS

Our aim is to develop a structure theory for pseudoeffect algebras. As intervals in po-groups served as prototypes, we ask, first, about group representations.

Now, even when assuming commutativity, it is, in spite of its importance for the foundations of quantum mechanics, an open problem how to characterize exactly those pseudoeffect algebras that are intervals of partially ordered groups. On the other hand, a certain Riesz property introduced in Section 3 is a sufficient condition; the aim of this section is to show that any pseudoeffect algebra that fulfils the Commutational Riesz Decomposition Property [see Definition 3.1(e)] is an interval pseudoeffect algebra.

We will use the so-called word technique, which was introduced by Baer (1949) and Wyler (1966). It has also been successfully applied to effect-algebras fulfilling the Riesz Interpolation Property (Ravindran, 1966) and to commutative

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BCK-algebras with the relative cancellation property (see Dvurečenskij and Pulmannová (2000), Chapter 5.2.5).

As a first step, we embed a given pseudoeffect algebra into a semigroup. The semigroup will then be extended to a *po*-group.

Definition 5.1. Let $(E; +, 0, 1)$ be a pseudoeffect algebra.

(i) A sequence $A = (a_1, \ldots, a_n)$ of finite, but nonzero, length with entries from E is called a *word* in E. We denote by $\mathcal{W}(E)$, the set of all words; that is,

$$\mathcal{W}(E) \stackrel{\text{def}}{=} \{(a_1, \ldots, a_n)\colon a_1, \ldots, a_n \in E, n \geq 1\}.$$

We define an *addition* in $\mathcal{W}(E)$ as the concatenation; that is,

$$+\colon \quad \mathcal{W}(E) \times \mathcal{W}(E) \to \mathcal{W}(E),$$
$$((a_1, \ldots, a_m), (b_1, \ldots, b_n)) \mapsto (a_1, \ldots, a_m, b_1, \ldots, b_n).$$

(ii) We call two words A and B of E *directly similar*, in symbols $A \sim B$, if one of it has the form $(a_1, \ldots, a_n)$, $n \geq 2$, and the other has the form $(a_1, \ldots, a_p + a_{p+1}, \ldots, a_n)$, $1 \leq p < n$.

We call two words A and B *similar*, in symbols $A \simeq B$, if there are words $A_0, \ldots, A_k$, $k \geq 0$, such that $A = A_0 \sim A_1 \sim \cdots \sim A_k = B$. In such a case, we say that A and B are *connected by a chain of length k.*

We set for $a_1, \ldots, a_n \in E$, $n \geq 1$,

$$[a_1, \ldots, a_n] \stackrel{\text{def}}{=} \{A \in \mathcal{W}(E)\colon A \simeq (a_1, \ldots, a_n)\},$$

and we put

$$\mathcal{C}(E) \stackrel{\text{def}}{=} \{[a_1, \ldots, a_n]\colon a_1, \ldots, a_n \in E, n \geq 1\}.$$

Lemma 5.2. *Let* $(E; +, 0, 1)$ *be a pseudoeffect algebra.*

(i) *Similarity in* $\mathcal{W}(E)$ *is an equivalence relation compatible with* $+$. *With* $+$ *as the induced relation,* $(\mathcal{C}(E); +)$ *is a semigroup with the neutral element* $[0]$.

(ii) *For* $a_1, \ldots, a_n, b \in E$, $n \geq 1$, $(a_1, \ldots, a_n) \simeq (b)$ *if and only if* $a_1 + \cdots + a_n$ *exists and equals* b.

Proof:

(i) By construction, $\simeq$ is an equivalence relation.

From $A_1 \simeq A$ and $B_1 \simeq B$, it follows that $A_1 + B_1 \simeq A + B$, so $+$ is definable in $\mathcal{C}(E)$.

As $\mathcal{W}(E)$ is associative, so is $\mathcal{C}(E)$, that is, $\mathcal{C}(E)$ is a semigroup. It has [0] as a neutral element, because, for example, $[a_1, \ldots, a_n] + [0] = [a_1, \ldots, a_n, 0] = [a_1, \ldots, a_n]$.

(ii) If for a word $(x_1, \ldots, x_m)$ the sum of its elements $x_1 + \cdots + x_m$ exists, the same is true for any word directly similar to $(x_1, \ldots, x_m)$, and the sums are equal. So the "only if" part follows by induction on the minimal length of a chain by which (b) and $(a_1, \ldots, a_n)$ are connected.
The "if" part is obvious. □

We note that (ii) of this lemma has been proved by Baer (Baer, 1949, Theorem 1) in a much more general context.

We will now prove the crucial lemma needed for the representation theorem. For the special notation used herein, see the paragraph preceding Lemma 3.9.

Lemma 5.3. *Let $(E; +, 0, 1)$ be a pseudoeffect algebra fulfilling (RDP$_1$). Let*

$$(a_1, \ldots, a_m) \simeq (b_1, \ldots, b_n),$$

where $m, n, \geq 1$. Then there are elements $d_{11}, \ldots, d_{mn} \in E$ such that

$$\begin{array}{ccccc} d_{11} & \cdots & d_{1n} & \to & a_1 \\ \vdots & & \vdots & & \vdots \\ d_{m1} & \cdots & d_{mn} & \to & a_m \\ \downarrow & & \downarrow & & \\ b_1 & \cdots & b_n & & \end{array}$$

and such that, for $1 \leq i < m, 1 \leq j < n$, we have

$$d_{i+1,j} + \cdots + d_{mj} \ \textbf{com} \ d_{i,j+1} + \cdots + d_{in}.$$

Proof: The proof is by induction on the minimal length k of a chain that connects $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_n)$.

If $k = 0$, we have $(a_1, \ldots, a_m) = (b_1, \ldots, b_n)$ and the elements in the scheme

$$\begin{array}{cccccc} a_1 & 0 & \cdots & 0 & \to & a_1 \\ 0 & a_2 & \cdots & 0 & \to & a_2 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_m & \to & a_m \\ \downarrow & \downarrow & & \downarrow & & \\ a_1 & a_2 & \cdots & a_m & & \end{array}$$

obviously fulfil the statements.

Suppose the statement holds for $k-1, k \geq 1$; we have to prove that it then holds also for k. So let $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_n)$ be connected by a chain of length k, say $A = A_0 \sim \cdots \sim A_k = B$.

There are two possibilities for how $A_k = B = (b_1, \ldots, b_n)$ is constructed from A_{k-1}.

1. Let $A_{k-1} = (b_1, \ldots, b_p^1, b_p^2, \ldots, b_n), 1 \leq p \leq n,$ and $b_p = b_p^1 + b_p^2$. Then, by hypothesis there are elements in E according to the scheme

$$\begin{array}{ccccccccc} d_{11} & \cdots & d_{1p}^1 & d_{1p}^2 & \cdots & d_{1n} & \to & a_1 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ d_{m1} & \cdots & d_{mp}^1 & d_{mp}^2 & \cdots & d_{mn} & \to & a_m \\ \downarrow & & \downarrow & \downarrow & & \downarrow & & \\ b_1 & \cdots & b_p^1 & b_p^2 & \cdots & b_n, & & \end{array} \tag{1}$$

such that the sums of elements placed below and right from any element fulfil the **com**-relation. Consider now the scheme

$$\begin{array}{ccccc} d_{11} & \cdots & d_{1p}^1 + d_{1p}^2 & \cdots & d_{1n} \\ \vdots & & \vdots & & \vdots \\ d_{m1} & \cdots & d_{mp}^1 + d_{mp}^2 & \cdots & d_{mm}. \end{array}$$

Here the lines still sum up to $a_1, \ldots, a_m$, and the unchanged columns clearly sum up as before to $b_1, \ldots, b_{p-1}, b_{p+1}, \ldots, b_n$. Because of the **com**-relations, we have that any element in (1) commutes with any other that is placed further up and right. So the pth column also adds up to what it should:

$$\begin{aligned} b_p = b_p^1 + b_p^2 &= d_{1p}^1 + \cdots + d_{mp}^1 + d_{1p}^2 + \cdots + d_{mp}^2 \\ &= d_{1p}^1 + d_{1p}^2 + d_{2p}^1 + \cdots + d_{mp}^1 + d_{2p}^2 + \cdots + d_{mp}^2 \\ &= \cdots \\ &= d_{1p}^1 + d_{1p}^2 + d_{2p}^1 + d_{2p}^2 + \cdots + d_{mp}^1 + d_{mp}^2. \end{aligned}$$

Now the only new conditions concerning the **com**-relation are the following, where $1 \leq j < m$:

$$d_{j+1,p}^2 + d_{j+1,p}^2 + \cdots + d_{mp}^1 + d_{mp}^2 \,\mathbf{com}\, d_{j,p+1} + \cdots + d_{jn}. \tag{2}$$

By the same reasoning as before, the first term in (2) is equal to $(d_{j+1,p}^1 + \cdots + d_{mp}^1) + (d_{j+1,p}^2 + \cdots + d_{mp}^2)$. Here, we know by hypothesis that both the first term in brackets and the second one fulfil the **com**-condition together with the second term in (2). Now by Lemma 3.2(i), (2) follows.

2. Let $A_{k-1} = (b_1, \ldots, b_p + b_{p+1}, \ldots, b_n)$, $1 \le p < n$. Then by hypothesis there are elements in E according to the scheme

$$\begin{array}{ccccccc} d_{11} & \cdots & e_1 & \cdots & d_{1n} & \to & a_1 \\ \vdots & & \vdots & & \vdots & & \vdots \\ d_{m1} & \cdots & e_m & \cdots & d_{mn} & \to & a_m \\ \downarrow & & \downarrow & & \downarrow & & \\ b_1 & \cdots & b_p + b_{p+1} & \cdots & b_n, & & \end{array}$$

such that the sums of elements placed below and right from any element fulfil the **com**-relation. Lemma 3.9 applied to

$$b_p + b_{p+1} = e_1 + \cdots + e_m$$

gives elements $d_{1p}, d_{1,p+1}, \ldots, d_{mp}, d_{m,p+1} \in E$, such that

$$\begin{array}{cccc} d_{1p} & d_{1,p+1} & \to & e_1 \\ \vdots & \vdots & & \vdots \\ d_{mp} & d_{m,p+1} & \to & e_m \\ \downarrow & \downarrow & & \\ b_p & b_{p+1} & & \end{array}$$

and for $1 \le j < m$,

$$d_{j+1,p} + \cdots + d_{mp} \ \textbf{com} \ d_{j,p+1}.$$

Then we have

$$\begin{array}{ccccccccc} d_{11} & \cdots & d_{1p} & d_{1,p+1} & \cdots & d_{1n} & \to & a_1 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ d_{m1} & \cdots & d_{mp} & d_{m,p+1} & \cdots & d_{mn} & \to & a_m \\ \downarrow & & \downarrow & \downarrow & & \downarrow & & \\ b_1 & \cdots & b_p & b_{p+1} & \cdots & b_n. & & \end{array}$$

As a sum when deleting any summand from it gets smaller, and as the **com**-relation is additive by Lemma 3.2(i), we see that the conditions concerning the **com**-relation are again preserved herein. □

With the help of Lemma 5.3, we will now check for the semigroup $\mathcal{C}(E)$ the properties that are necessary and sufficient for extending it to a *po*-group whose positive cone it is. The extension itself is described in the subsequent Definition 5.5 and Lemma 5.6.

Lemma 5.4. *Let* $(E; +, 0, 1)$ *be a pseudoeffect algebra fulfilling (RDP*$_1$*). Then* $\mathcal{C}(E)$ *is a semigroup such that the following hold:*

(i) $[0]$ *is a neutral element.*
(ii) *Let* $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}(E)$. *From* $\mathfrak{a} + \mathfrak{b} = [0]$, *it follows that* $\mathfrak{a} = \mathfrak{b} = [0]$.

(iii) *Let* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{C}(E)$. *From* $\mathfrak{a} + \mathfrak{b} = \mathfrak{a} + \mathfrak{c}$, *it follows that* $\mathfrak{b} = \mathfrak{c}$; *from* $\mathfrak{b} + \mathfrak{a} = \mathfrak{c} + \mathfrak{a}$, *it follows that* $\mathfrak{b} = \mathfrak{c}$.

(iv) *For any pair* $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}(E)$, *there is a* $\mathfrak{c} \in \mathcal{C}(E)$ *such that* $\mathfrak{a} + \mathfrak{b} = \mathfrak{c} + \mathfrak{a}$, *and there is a* $\mathfrak{d} \in \mathcal{C}(E)$ *such that* $\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{d}$.

Proof: That $\mathcal{C}(E)$ is a semigroup fulfilling (i) has been proved in Lemma 5.2(i). Part (ii) follows from Lemma 5.2(ii) and Lemma 1.4(ii).

For part (iii), we may suppose $\mathfrak{a} = [a]$, and let $\mathfrak{b} = [b_1, \ldots, b_m]$, $\mathfrak{c} = [c_1, \ldots, c_n]$. So we will show that $(a, b_1, \ldots, b_m) \simeq (a, c_1, \ldots, c_n)$ implies $(b_1, \ldots, b_m) \simeq (c_1, \ldots, c_n)$; then the first part will follow, and the second one is proved analogously.

By our assumption, there are, by Lemma 5.3, elements in E according to the following scheme:

$$\begin{array}{ccccc} d & d_1 & \cdots & d_n & \to & a \\ e_1 & e_{11} & \cdots & e_{1n} & \to & b_1 \\ \vdots & \vdots & & \vdots & & \vdots \\ e_m & e_{m1} & \cdots & e_{mn} & \to & b_m \\ \downarrow & \downarrow & & \downarrow & & \\ a & c_1 & & c_n, & & \end{array}$$

where for $1 \le j < i \le m$, $1 \le k \le n$, we have $e_i + e_{jk} = e_{jk} + e_i$, and for $1 \le k < i \le m$, $1 \le j < l \le n$, we have $e_{ij} + e_{kl} = e_{kl} + e_{ij}$, and for $1 \le i \le m$, $1 \le j < k \le n$, we have $e_{ij} + d_k = d_k + e_{ij}$. Now, $a = d + d_1 + \cdots + d_n = d + e_1 + \cdots + e_m$ implies $d_1 + \cdots + d_n = e_1 + \cdots + e_m$. Using this and the commutativity relations, we get

$$\begin{aligned} (b_1, \ldots, b_m) &\simeq (e_1, e_{11}, \ldots, e_{1n}, \ldots, e_m, e_{m1}, \ldots, e_{mn}) \\ &\simeq (e_1, \ldots, e_m, e_{11}, \ldots, e_{m1}, \ldots, e_{1n}, \ldots, e_{mn}) \\ &\simeq (d_1, \ldots, d_n, e_{11}, \ldots, e_{m1}, \ldots, e_{1n}, \ldots, e_{mn}) \\ &\simeq (d_1, e_{11}, \ldots, e_{m1}, \ldots, d_n, e_{1n}, \ldots, e_{mn}) \\ &\simeq (c_1, \ldots, c_n). \end{aligned}$$

In case of (iv), we will prove the first part only; the second is shown analogously. Moreover, we suppose that $\mathfrak{a} = [a]$ and $\mathfrak{b} = [b]$; the claim then follows by an easy induction argument. So what we will show is that there is a word $\mathfrak{c}$ such that $(a, b) = \mathfrak{c} + (a)$.

By (RDP$_1$) we have for some $d_1, \ldots, d_4 \in E$,

$$\begin{array}{cccc} d_1 & d_2 & \to & a \\ d_3 & d_4 & \to & a^{\sim} \\ \downarrow & \downarrow & & \\ b^{-} & b. & & \end{array}$$

So there are elements $d_2', d_4' \in E$ such that $(a, b) = (d_1 + d_2, d_2 + d_4) \sim (d_2' + d_1, d_2, d_4) \simeq (d_2', a + d_4) = (d_2', d_4' + a) \sim (d_2', d_4') + (a)$, where we used the fact that because of $d_4 \le a^{\sim}$ and Lemma 1.6(iv), $a + d_4$ exists. □

Definition 5.5. Let $(E; +, 0, 1)$ be a pseudoeffect algebra fulfilling (RDP$_1$). In view of Lemma 5.4, we may define, for any $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}(E)$, $\mathfrak{a}_{\mathfrak{b}}$ to be the unique element such that $\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}_{\mathfrak{b}}$.

Define for $\mathcal{C}(E) \times \mathcal{C}(E)$ the binary operation $+$ by

$$(\mathfrak{a}, \mathfrak{b}) + (\mathfrak{c}, \mathfrak{d}) \stackrel{\text{def}}{=} (\mathfrak{a} + \mathfrak{c}_{\mathfrak{b}}, \mathfrak{d} + \mathfrak{b})$$

for $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \mathcal{C}(E)$.

Define for $\mathcal{C}(E) \times \mathcal{C}(E)$

$$(\mathfrak{a}, \mathfrak{b}) \approx (\mathfrak{c}, \mathfrak{d}) \stackrel{\text{def}}{\leftrightarrow} \mathfrak{a} + \mathfrak{d}_{\mathfrak{b}} = \mathfrak{c} + \mathfrak{b},$$

$$\langle \mathfrak{a}, \mathfrak{b} \rangle \stackrel{\text{def}}{=} \{(\mathfrak{c}, \mathfrak{d}) : (\mathfrak{c}, \mathfrak{d}) \approx (\mathfrak{a}, \mathfrak{b})\},$$

$$\mathcal{G}(E) \stackrel{\text{def}}{=} \{\langle \mathfrak{a}, \mathfrak{b} \rangle : \mathfrak{a}, \mathfrak{b} \in \mathcal{C}(E)\},$$

$$\mathcal{G}(E)^+ \stackrel{\text{def}}{=} \{\langle \mathfrak{a}, [0] \rangle : \mathfrak{a} \in \mathcal{C}(E)\},$$

$$\iota_E\colon \quad E \to \mathcal{G}(E), \quad a \mapsto \langle [a], [0] \rangle.$$

Lemma 5.6. *Let $(E; +, 0, 1)$ be a pseudoeffect algebra fulfilling (RDP$_1$). Then the relation $\approx$ on $\mathcal{C}(E) \times \mathcal{C}(E)$ is an equivalence relation that is compatible with $+$. With $+$ as the derived operation and $\mathcal{G}(E)^+$ as the positive cone, $(\mathcal{G}(E); +, \le)$ is a po-group. We have $\mathcal{G}(E) = \mathcal{G}(E)^+ - \mathcal{G}(E)^+ = -\mathcal{G}(E)^+ + \mathcal{G}(E)^+$.*

Now, $\varepsilon\colon \mathcal{C}(E) \to \mathcal{G}(E)$, $\mathfrak{a} \mapsto \langle \mathfrak{a}, [0] \rangle$ establishes a semigroup isomorphism between $\mathcal{C}(E)$ and $\mathcal{G}(E)^+$.

Proof: This lemma is a consequence of Lemma 5.4. For the details of the proof, see Fuchs (1965), Theorem II.4. □

We arrive at our main theorem.

Theorem 5.7. *Let $(E; +, 0, 1)$ be a pseudoeffect algebra fulfilling (RDP$_1$). Then $\iota_E\colon E \to \mathcal{G}(E)$, $a \mapsto \langle [a], [0] \rangle$ determines an isomorphism between $(E; +, 0, 1)$ and $(\Gamma(\mathcal{G}(E), \langle [1], [0] \rangle); +, \langle [0], [0] \rangle, \langle [1], [0] \rangle)$, where $\langle [1], [0] \rangle$ is a strong unit of $\mathcal{G}(E)$.*

In particular, E is an interval pseudoeffect algebra.

Proof: ι_E is injective. Indeed, for $a, b \in E$, $\langle [a], [0] \rangle = \langle [b], [0] \rangle$ implies $[a] = [b]$ by the injectivity of ε from Lemma 5.6, and this implies $a = b$ by Lemma 5.2(ii).

The image of ι_E is $\Gamma(\mathcal{G}(E), \langle[1], [0]\rangle)$. Indeed, any positive element of $\mathcal{G}(E)$ is of the form $\langle\mathfrak{a}, [0]\rangle$ for some $\mathfrak{a} \in \mathcal{C}(E)$ by the surjectivity of ε from Lemma 5.6. Now, $\langle\mathfrak{a}, [0]\rangle \le \langle[1], [0]\rangle$ means for some other positive element $\langle\mathfrak{b}, [0]\rangle$, where $\mathfrak{b} \in \mathcal{C}(E)$, that $\langle\mathfrak{a}, [0]\rangle + \langle\mathfrak{b}, [0]\rangle = \langle[1], [0]\rangle$. But then $\mathfrak{a} + \mathfrak{b} = [1]$, because, by Lemma 5.6, ε is a semigroup isomorphism. It follows by Lemma 5.2(ii) that $\mathfrak{a} = [a]$, $\mathfrak{b} = [b]$ for some $a, b \in E$, that is, $\langle\mathfrak{a}, [0]\rangle = \iota_E(a)$.

Now, $\iota_E\colon E \to \Gamma(\mathcal{G}(E), \langle[1], [0]\rangle)$ is an isomorphism with respect to $+$. Indeed, for $a, b \in E$, $a + b$ exists in E and equals c iff $(a, b) \sim (c)$ iff $[c] = [a] + [b]$ iff $\langle[c], [0]\rangle = \langle[a], [0]\rangle + \langle[b], [0]\rangle$ iff $\iota_E(c) = \iota_E(a) + \iota_E(b)$. Here we used again the fact that ε from Lemma 5.6 is a semigroup isomorphism.

It remains to show that $\langle[1], [0]\rangle$ is a strong unit of $\mathcal{G}(E)$. Because $\mathcal{G}(E) = \mathcal{G}(E)^+ - \mathcal{G}(E)^+$ holds and because $\mathcal{G}(E)^+$ is isomorphic to $\mathcal{C}(E)$, it is sufficient to show that any $\mathfrak{a} = [a_1, \dots, a_n] \in \mathcal{C}(E)$ lies below a multiple of [1]. By Lemma 5.4(iv), we conclude that

$$\big[\underbrace{1, \dots, 1}_{n \text{ times}}\big] = \big[a_1, a_1^{\sim}, \dots, a_n, a_n^{\sim}\big] = [a_1, \dots, a_n] + \mathfrak{b} \quad \text{for some } \mathfrak{b} \in \mathcal{C}(E),$$

which means

$$\mathfrak{a} \le \underbrace{[1] + \cdots + [1]}_{n \text{ times}}. \qquad \square$$

From now on, we will consider $\mathcal{C}(E)$ as a subset of $\mathcal{G}(E)$, thus identifying $\mathfrak{a} \in \mathcal{C}(E)$ with $\langle\mathfrak{a}, [0]\rangle \in \mathcal{G}(E)$; in particular, we write [0] and [1] instead of $\langle[0], [0]\rangle$ and $\langle[1], [0]\rangle$, respectively. Note that we then have $\langle\mathfrak{a}, \mathfrak{b}\rangle = \mathfrak{a} - \mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}(E)$.

6. PROPERTIES OF A PSEUDOEFFECT ALGEBRA AND ITS REPRESENTING GROUP

We check properties that are preserved from a *po*-group G with positive element u in the pseudoeffect algebra $\Gamma(G, u)$ (see Definition 2.1) and from a pseudoeffect algebra E fulfilling (RDP$_1$) in the representing group $\mathcal{G}(E)$ (see Theorem 5.7).

6.1. Lattice order and Riesz properties

Proposition 6.1. *Let G be a po-group with positive element u.*

(i) *If then G is an ℓ-group, then $\Gamma(G, u)$ is also lattice-ordered.*
(ii) *Let G be directed. If G has one of the properties (RIP), (RDP$_0$), (RDP), (RDP$_1$), or (RDP$_2$), then $\Gamma(G, u)$ has the equally denoted property.*

In the other direction, we have the following.

Proposition 6.2. *Let E be a pseudoeffect algebra fulfilling (RDP$_1$). Then also $\mathcal{G}(E)$ fulfils (RDP$_1$).*

Proof: Let $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{G}(E)$ be positive elements of $\mathcal{G}(E)$ such that $\mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{b}_1 + \mathfrak{b}_2$. Let $\mathfrak{a}_1 = [a^1_1, \dots, a^1_k]$, $\mathfrak{a}_2 = [a^2_1, \dots, a^2_l]$, $\mathfrak{b}_1 = [b^1_1, \dots, b^1_m]$, $\mathfrak{b}_2 = [b^2_1, \dots, b^2_n]$, where $a^1_1, \dots, b^2_n \in E$. By Lemma 5.3, there are $d^1_{11}, \dots, d^4_{ln} \in E$ such that

$$\begin{array}{ccccccccc}
d^1_{11} & \cdots & d^1_{1m} & d^2_{11} & \cdots & d^2_{1n} & \to & a^1_1 \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
d^1_{k1} & \cdots & d^1_{km} & d^2_{k1} & \cdots & d^2_{kn} & \to & a^1_k \\
d^3_{11} & \cdots & d^3_{1m} & d^4_{11} & \cdots & d^2_{1n} & \to & a^2_1 \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
d^3_{l1} & \cdots & d^3_{lm} & d^4_{l1} & \cdots & d^4_{ln} & \to & a^2_l \\
\downarrow & & \downarrow & \downarrow & & \downarrow & & \\
b^1_1 & \cdots & b^1_m & b^2_1 & \cdots & b^2_n & &
\end{array}$$

and such that every two elements in this diagram fulfil the **com**-condition if one of them is placed further up and further right than the other one.

Define now $\mathfrak{d}_1 = [d^1_{11}, \dots, d^1_{1m}, \dots, d^1_{k1}, \dots, d^1_{km}]$ and in a similar way also $\mathfrak{d}_2, \mathfrak{d}_3, \mathfrak{d}_4$. Because of the commutativity conditions, we have then

$$\begin{array}{cccc}
\mathfrak{d}_1 & \mathfrak{d}_2 & \to & \mathfrak{a}_1 \\
\mathfrak{d}_3 & \mathfrak{d}_4 & \to & \mathfrak{a}_2 \\
\downarrow & \downarrow & & \\
\mathfrak{b}_1 & \mathfrak{b}_2. & &
\end{array}$$

Now let $0 \le \mathfrak{x} \le \mathfrak{d}_2$ and $0 \le \mathfrak{y} \le \mathfrak{d}_3$. So $\mathfrak{x} + \mathfrak{x}' = \mathfrak{d}_2$ for some $\mathfrak{x}' \in \mathcal{C}(E)$, and we may apply Lemma 5.3 to this equation to conclude that $\mathfrak{x}$ is representable as a word such that every element in it lies below some element in $\mathfrak{d}_2$. The same applies to $\mathfrak{y}$ and $\mathfrak{d}_3$.

Now for every $x, y \in E$ such that x lies below an element occurring in the word $\mathfrak{d}_2$ and y lies below an element occurring in $\mathfrak{d}_3$, we have $x + y = y + x$. We conclude that $\mathfrak{x}$ and $\mathfrak{y}$ commute. So we have proved $\mathfrak{d}_2$ **com** $\mathfrak{d}_3$, which means that (RDP$_1$) holds in $\mathcal{G}(E)$. $\square$

Proposition 6.3. *Let E be a pseudoeffect algebra fulfilling (RDP$_1$). The embedding $\iota_E \colon E \mapsto \mathcal{G}(E)$, $a \mapsto \langle [a], [0] \rangle$ preserves infima and suprema.*

Proof: Let $a, b \in E$ such that $a \wedge b$ exists. As ι_E is order-preserving, $\iota_E(a \wedge b) \le \iota_E(a), \iota_E(b)$, that is, under the above identification of $\mathcal{C}(E)$ and $\mathcal{G}(E)^+$, $[a \wedge b] \le [a], [b]$. Suppose an element of $\mathcal{G}(E)$ to be a lower bound of $[a]$ and $[b]$, that is, $\langle \mathfrak{x}, \mathfrak{y} \rangle \le [a], [b]$ for some $\mathfrak{x}, \mathfrak{y} \in \mathcal{C}(E)$.

Because by Propositions 6.2 and 4.2(i), (RIP) holds in $\mathcal{G}(E)$, we may conclude from $\mathfrak{x}, \mathfrak{y} \leq [a] + \mathfrak{y}, [b] + \mathfrak{n}$ that, for some $\mathfrak{c} \in \mathcal{G}(E)^+$, we have $\mathfrak{x} \leq \mathfrak{c} + \mathfrak{y} \leq [a] + \mathfrak{y}, [b] + \mathfrak{y}$, which means $\langle \mathfrak{x}, \mathfrak{y} \rangle \leq \mathfrak{c} \leq [a], [b]$.

As $\mathfrak{c} = [c]$ for some $c \in E$, we have from $c \leq a, b$ that $c \leq a \wedge b$, and so $\langle \mathfrak{x}, \mathfrak{y} \rangle \leq [c] \leq [a \wedge b]$. It follows that $[a \wedge b] = [a] \wedge [b]$. So we have shown that ι_E preserves infima.

Let now $a, b \in E$ such that $a \vee b$ exists. Choose $\overline{a}, \overline{b} \in E$ such that $a + \overline{a} = b + \overline{b} = a \vee b$; then $\iota_E(\overline{a}) = -\iota_E(a) + \iota_E(a \vee b)$ and $\iota_E(\overline{b}) = -\iota_E(b) + \iota_E(a \vee b)$. We claim $\overline{a} \wedge \overline{b} = 0$. Indeed, suppose $x \leq \overline{a}, \overline{b}$ for some $x \in E$; then, by Lemma 1.6(v), we conclude that $a + x, b + x \leq a \vee b = y + x$ for some $y \in E$, so $a, b \leq y$ and $a \vee b \leq y$; it follows that $(a \vee b) + x \leq y + x = a \vee b$ and finally $x = 0$. So

$$0 = \iota_E(\overline{a} \wedge \overline{b}) = \iota_E(\overline{a}) \wedge \iota_E(\overline{b}) = [-\iota_E(a) + \iota_E(a \vee b)] \wedge [-\iota_E(b) + \iota_E(a \vee b)]$$

and it follows that $\iota_E(a \vee b) = \iota_E(a) \vee \iota_E(b)$. So ι_E also preserves suprema. □

Proposition 6.4. *Let E be a pseudoeffect algebra fulfilling (RDP$_1$). Then the following statements are equivalent.*

(α) *E fulfils (RDP$_2$).*
(β) *E is lattice-orderded.*
(γ) *$\mathcal{G}(E)$ fulfils (RDP$_2$).*
(δ) *$\mathcal{G}(E)$ is an ℓ-group.*

Proof: As E fulfils (RDP$_0$), we have by Proposition 3.3(ii) that E fulfils (RDP$_2$) iff E is lattice-ordered. By Proposition 4.2(ii), we have that $\mathcal{G}(E)$ fulfils (RDP$_2$) iff $\mathcal{G}(E)$ is lattice-ordered. By Proposition 6.1(i), if $\mathcal{G}(E)$ fulfils (RDP$_2$), then so does E.

Now let E fulfil (RDP$_2$). Given $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{G}(E)$ such that $\mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{b}_1 + \mathfrak{b}_2$, we construct in the same manner as in the proof of Proposition 6.2 elements $\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3, \mathfrak{d}_4$, with the only difference that now the infimum of any element of E occuring in $\mathfrak{d}_2$ and any element occuring in $\mathfrak{d}_3$ is required to be 0. This is easily done by modifying the proofs of the Lemmas 3.2(i), 3.9, and 5.3, replacing the **com**- by the zero-infimum condition.

Because by Proposition 6.3 the relation of having a zero infimum is preserved from E in $\mathcal{G}(E)$, and because it is in additive in ℓ-groups, we conclude that $\mathfrak{d}_2 \wedge \mathfrak{d}_3 = 0$. So (RDP$_2$) holds in $\mathcal{G}(E)$. □

We conclude with an example showing that, given a unital group (G, u), if (RIP) holds in $\Gamma(G, u)$, it does not necessarily follow that (RIP) also holds in G.

Example 6.5. Let G be the product of the group $\mathbb{Z}$ of integers and the group $\mathbb{Z}_2 = \mathbb{Z}$ mod 2, that is, $G = \mathbb{Z} \times \mathbb{Z}_2$. Define, for $a_1, a_2, b_1, b_2 \in \mathbb{Z}$,

$$(a_1, a_2) \leq (b_1, b_2) \quad \text{iff } a_1 < b_1 \text{ or } a_1 = b_1 \text{ and } a_2 = b_2.$$

Then G is a *po*-group with strong unit $u = (2, 0)$, and $\Gamma(G, u)$ is the diamond from Example 3.7. As was proved there, $\Gamma(G, u)$ fulfils (RIP), but G does not, as is seen, e.g., from the inequality $(0, 0), (0, 1) \le (1, 0), (1, 1)$, which has no interpolant.

6.2. Commutativity

Proposition 6.6. *Let G be a po-group with positive element u. If G is Abelian, then $\Gamma(G, u)$ is also commutative.*

Proposition 6.7. *Let E be a pseudoeffect algebra fulfilling (RDP_1). If E is commutative, then $\mathcal{G}(E)$ is also commutative.*

Proof: Suppose first that $\mathfrak{a}, \mathfrak{b} \in \mathcal{G}(E)^+$. By Lemma 5.4(iv), there is a unique $\mathfrak{c} \in \mathcal{G}(E)^+$ such that $\mathfrak{a} + \mathfrak{b} = \mathfrak{c} + \mathfrak{a}$. By the proof of that lemma, we see that if E is commutative, then $\mathfrak{c} = \mathfrak{b}$.

Because by Lemma 5.6, $\mathcal{G}(E)^+$ generates $\mathcal{G}(E)$, we conclude that $\mathcal{G}(E)$ is commutative. □

6.3. Linearity

Proposition 6.8. *Let G be a po-group with positive element u. If G is linearly ordered, then $\Gamma(G, u)$ is also linearly ordered.*

Proposition 6.9. *Let E be a pseudoeffect algebra fulfilling (RDP_1). If E is linearly ordered, then $\mathcal{G}(E)$ is also linearly ordered.*

Proof: We prove first that every two elements from $\mathcal{G}(E)^+$ are comparable. If $0 \le \mathfrak{a}, \mathfrak{b} \le [1]$, this follows by assumption. Suppose now that every two elements from $\mathcal{G}(E)^+$ each of which is the sum of not more than $n - 1$ elements from the unit interval are comparable, where $n \ge 2$, and let $\mathfrak{a} = [a_1, \dots, a_n]$, $\mathfrak{b} = [b_1, \dots, b_n]$, where $a_1, \dots, a_n, b_1, \dots, b_n \in E$. To see that then $\mathfrak{a}$ and $\mathfrak{b}$ are also comparable, we check the case $a_1 \le b_1$ and $[a_2, \dots, a_n] \ge [b_2, \dots, b_n]$. Choose $\mathfrak{r}, \mathfrak{s} \in \mathcal{G}(E)^+$ such that $[b_1] = [a_1] + \mathfrak{r}$ and $[a_2, \dots, a_n] = \mathfrak{s} + [b_2, \dots, b_n]$. Then $\mathfrak{r} \le [1]$, and from $\mathfrak{s} \le [a_2, \dots, a_n]$, we see by ($RDP_1$), which holds in $\mathcal{G}(E)$ by Proposition 6.4(i), that $\mathfrak{s}$ is the sum of $n - 1$ elements of the unit interval. So $\mathfrak{r}$ and $\mathfrak{s}$ are comparable, which menas that $\mathfrak{a} = [a_1] + \mathfrak{s} + [b_2, \dots, b_n]$ and $\mathfrak{b} = [a_1] + \mathfrak{r} + [b_2, \dots, b_n]$ also are. The claim follows, by induction.

Now suppose two elements from $\mathcal{G}(E)$ to be given. Because by Lemma 5.6, $\mathcal{G}(E) = \mathcal{G}(E)^+ - \mathcal{G}(E)^+ = -\mathcal{G}(E)^+ + \mathcal{G}(E)^+$, we may suppose these elements to have the form $\mathfrak{a} - \mathfrak{b}$ and $-\mathfrak{c} + \mathfrak{d}$, respectively, for some $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \mathcal{G}(E)^+$. But

$\mathfrak{a} - \mathfrak{b}$ and $-\mathfrak{c} + \mathfrak{d}$ are comparable iff $\mathfrak{c} + \mathfrak{a}$ and $\mathfrak{d} + \mathfrak{b}$ are comparable, and this is the case. □

6.4. Archimedeanicity

We introduce archimedeanicity for pseudoeffect algebras.

Definition 6.10. A pseudoeffect algebra $(E; +, 0, 1)$ is said to be *Archimedean* if, for every $a \in E$, the existence of na for every $n \geq 1$ implies $a = 0$.

We note that the algebra from Example 2.3 is non-Archimedian: There, we have that $n\,(0, 0, 1)$ is defined for every $n \geq 1$.

But we see that the following holds.

Proposition 6.11. *Let E be a σ-complete lattice pseudoeffect algebra. Then E is Archimedean.*

Proof: Suppose $a \in E$ and na to be defined for any $n \geq 1$. As in the proof of Proposition 3.11, we conclude $(\bigvee_m ma) + a = \bigvee_m (ma + a) = \bigvee_m ma$, hence $a = 0$. □

For groups, we shall use the following definition of Archimedeanicity.

Definition 6.12.[2] A *po*-group $(G; +, \leq)$ is said to be *Archimedean* if, for every $a, b \in G$, $na \leq b$ for every $n \geq 1$ implies $a \leq 0$.

Proposition 6.13. *Let (G, u) be a unital po-group such that G is Archimedean. Then $\Gamma(G, u)$ is Archimedean and commutative.*

Proof: That under the assumptions on G, $\Gamma(G, u)$ is Archimedean is obvious. Moreover, because G possesses a strong unit, it is directed, and in this case we know by Fuchs (1963), Chapter V.1.G and Corollary V.20, that G is Abelian; so $\Gamma(G, u)$ is commutative. □

We do not yet know if in general Archimedeanicity of a pseudoeffect algebra E fulfilling (RDP_1) implies the same property for the representing group $\mathcal{G}(E)$, and so by Proposition 6.13, also commutativity. But we note the following.

Proposition 6.14. *Let E be a linear, Archimedean pseudoeffect algebra fulfilling (RDP_1). Then E is commutative, and $\mathcal{G}(E)$ is Archimedean and Abelian.*

[2]The property defined here is called by some authors *completely integrally closed*, for example, in Fuchs (1963).

Proof: By Proposition 6.9. $\mathcal{G}(E)$ is linearly ordered.

Let $a, b \in \mathcal{G}(E)$ and $na \leq b$ for all $n \geq 1$. We have to show that $a \leq 0$. But if $a < 0$, we are done; so suppose $a \geq 0$, in which case we have to prove $a = 0$.

If $b \leq [1]$, $a = 0$ follows from the Archimedeanicity of E. Let us assume that $a = 0$ follows whenever b is the sum of $k - 1$ elements of the unit interval, where $k \geq 2$. Suppose $na < b = b_1 + \cdots + b_k$ for all n. Then $na - b_k < b_1 + \cdots + b_{k-1}$, from which by assumption it follows that $na - b_k \leq 0$; this means $na \leq b_k$ for n, so we have, again by assumption, $a = 0$.

So $\mathcal{G}(E)$ is Archimedean. As it is a directed group, we again know by Fuchs (1963), Chapter V.1.G and Corollary V.20, that it is Abelian. So E is commutative. □

7. CATEGORICAL ISOMORPHISM OF PSEUDOEFFECT ALGEBRAS AND *po*-GROUPS WITH THE RIESZ PROPERTY

We shall show that pseudoeffect algebras and unital *po*-groups, both fulfilling (RDP$_1$), are categorically equivalent.

Definition 7.1. Let $(E; +, 0, 1)$ be a pseudoeffect algebra. Then a pair $((G, u), \iota_E)$ of a unital *po*-group group (G, u) and a homomorphism $\iota_E\colon (E; +, 0, 1) \to (G; +, \leq, u)$ is called a *universal group* for E if, for every homomorphism $h : (E; +, 0, 1) \to (H; +, \leq, v)$ of E into a unital *po*-group (H, v) there is a homomorphism $k : (G; +, \leq, u) \longrightarrow (H; +, \leq, v)$ such that $h = k \circ \iota_E$.

Theorem 7.2. *Let $(E; +, 0, 1)$ be a pseudoeffect algebra fulfilling (RDP$_1$).*

(i) *$((\mathcal{G}(E), [1]), \iota_E)$, where ι_E has been defined in Definition 5.5, is a universal group for E.*

(ii) *Let $\varphi\colon (E; +, 0_E, 1_E) \to (F; +, 0_F, 1_F)$ be a homomorphism of E into another pseudoeffect algebra fulfilling (RDP$_1$). Then there is a unique homomorphism $\psi\colon (\mathcal{G}(E), [1_E] \to (G(F), [1_F])$ of unital po-groups such that the following diagram commutes.*

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & F \\ \iota_E\downarrow & & \iota_F\downarrow \\ \mathcal{G}(E) & \xrightarrow{\psi} & \mathcal{G}(F) \end{array}$$

Proof: (i) Let H be a *po*-group, v be a strong unit of H, and $h\colon (E; +, 0, 1) \to (H; +, \leq, v)$ be a homomorphism; that is, $h(E) \subseteq H^+$, $+$ is preserved, $h(0)$ is the group zero, and $h(1) = v$.

As ι_E: $E \to \Gamma((\mathcal{G}(E), [1]), a \mapsto [a]$, is bijective by Theorem 5.7, we may define

$$k^1\colon\ \Gamma(\mathcal{G}(E), [1]) \to H, \quad [a] \mapsto h(a).$$

Obviously, $k^1 \circ \iota_E = h$, k^1 preserves $+$, and $k^1([1]) = h(1) = v$.

We may extend k^1 to $\mathcal{G}(E)^+$ by requiring

$$k^+\colon\ \mathcal{G}(E)^+ \to H, [a_1, \ldots, a_n] \mapsto h(a_1) + \cdots + h(a_n).$$

This is possible because directly similar words $(x_1, \ldots, x_m)$ and $(x_1, \ldots, x_p + x_{p+1}, \ldots, x_m)$, where $x_1, \ldots, x_m \in E$, $m \geq 1$, and $1 \leq p < m$, are both mapped to $h(x_1) + \cdots + h(x_m)$.

By construction, k^+ preserves $+$ also.

We may further extend k^+ to $\mathcal{G}(E)$ by requiring

$$k\colon\ \mathcal{G}(E) \to H, \quad \langle \mathfrak{a}, \mathfrak{b}\rangle \mapsto k^+(\mathfrak{a}) - k^+(\mathfrak{b})$$

For, let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \mathcal{C}(E)$ and suppose $\langle \mathfrak{a}, \mathfrak{b}\rangle = \langle \mathfrak{c}, \mathfrak{d}\rangle$. That means, in view of Definition 5.5, $(\mathfrak{a}, \mathfrak{b}) \approx (\mathfrak{c}, \mathfrak{d})$, that is, $\mathfrak{a} + \mathfrak{d}_\mathfrak{b} = \mathfrak{c} + \mathfrak{b}$, where $\mathfrak{d}_\mathfrak{b}$ is subject to the condition $\mathfrak{b} + \mathfrak{d}_\mathfrak{b} = \mathfrak{d} + \mathfrak{b}$. So we have $k^+(\mathfrak{a}) + k^+(\mathfrak{d}_\mathfrak{b}) = k^+(\mathfrak{c}) + k^+(\mathfrak{b})$, and $k^+(\mathfrak{b}) + k^+(\mathfrak{d}_\mathfrak{b}) = k^+(\mathfrak{d}) + k^+(\mathfrak{b})$. Both equations combined give $-k^+(\mathfrak{b}) + k^+(\mathfrak{d}) = k^+(\mathfrak{d}_\mathfrak{b}) - k^+(\mathfrak{b}) = -k^+(\mathfrak{a}) + k^+(\mathfrak{c})$, that is, $k^+(\mathfrak{a}) - k^+(\mathfrak{b}) = k^+(\mathfrak{c}) - k^+(\mathfrak{d})$.

It remains to show that k is a homomorphism. We first check that k preserves $+$. As k^+ does so, this is the case the for positive elements. Now let again $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \mathcal{C}(E)$. Then we have

$$\begin{aligned} k(\langle \mathfrak{a}, \mathfrak{b}\rangle + \langle \mathfrak{c}, \mathfrak{d}\rangle) &= k(\langle \mathfrak{a} + \mathfrak{c}_\mathfrak{b}, \mathfrak{d} + \mathfrak{b}\rangle) \\ &= k(\mathfrak{a}) + k(\mathfrak{c}_\mathfrak{b}) - k(\mathfrak{b}) - k(\mathfrak{d}) \\ &= k(\mathfrak{a}) - k(\mathfrak{b}) + k(\mathfrak{c}) - k(\mathfrak{d}) \\ &= k(\langle \mathfrak{a}, \mathfrak{b}\rangle) + k(\langle \mathfrak{c}, \mathfrak{d}\rangle). \end{aligned}$$

Second, k preserves the order; for, $h(E) \subseteq H^+$, so $k(\mathcal{G}(E)^+) = k^+(\mathcal{G}(E)^+) \subseteq H^+$. Finally, we have $k([1]) = k^1([1]) = v$.

(ii) As $\iota_F \circ \varphi$: $E \to \mathcal{G}(F)$ is a homomorphism of E into $\mathcal{G}(F)$, the existence of a function ψ: $(\mathcal{G}(E), [1_E]) \to (\mathcal{G}(F), [1_F])$ such that $\iota_F \circ \varphi = \psi \circ \iota_E$ follows from (i). The uniqueness is obvious from the proof of (i). $\square$

Definition 7.3. Let $\mathcal{RPEA}$ denote the category whose objects are the pseudoeffect algebras fulfilling (RDP_1) and whose morphisms are the homomorphisms of these structures.

Let $\mathcal{RPOG}$ denote the category whose objects are the unital *po*-groups fulfilling (RDP_1) and whose morphisms are the homomorphisms of these structures.

Define Γ to be the functor from $\mathcal{RPOG}$ to $\mathcal{RPEA}$ that maps an object (G, u) from $\mathcal{RPOG}$ to $\Gamma(G, u)$ according to Definition 2.1 and maps a morphism $\psi\colon (G, u) \to (H, v)$ from $\mathcal{RPOG}$ to its restriction to $\Gamma(G, u)$.

Define Δ to be the functor from $\mathcal{RPEA}$ to $\mathcal{RPOG}$ that maps an object E from $\mathcal{RPEA}$ to $(\mathcal{G}(E), [1])$ and maps a morphism $\varphi\colon E \to F$ from $\mathcal{RPEA}$ to the morphism $\psi\colon \mathcal{G}(E) \to \mathcal{G}(F)$ subject to the condition $\iota_F \circ \varphi = \psi \circ \iota_E$.

That Γ and Δ are well-defined as functors follows by construction.

Theorem 7.4. *The functors $\Delta\colon \mathcal{RPEA} \to \mathcal{RPOG}$ and $\Gamma\colon \mathcal{RPOG} \to \mathcal{RPEA}$ form an equivalence of categories, that is, $\Gamma \circ \Delta$ and $\Delta \circ \Gamma$ are naturally equivalent to the identity functors of $\mathcal{RPEA}$ and $\mathcal{RPOG}$, respectively.*

Proof: We have to show that any object E from $\mathcal{RPEA}$ is isomorphic to $\Gamma(\Delta(E))$ and that any G from $\mathcal{RPOG}$ is isomorphic to $\Delta(\Gamma(G))$.

So let E be a pseudoeffect algebra fulfilling (RDP_1). By Theorem 5.7, $\Gamma(\Delta(E)) = \Gamma(\mathcal{G}(E), [1]) \cong E$.

Let (G, u) be a unital *po*-group that fulfils (RDP_1). We have to check $(\mathcal{G}(\Gamma(G, u)), [u]) \cong (G, u)$. Now, by Theorem 5.7, the unit intervals of $\mathcal{G}(\Gamma(G, u))$ and of G are isomorphic; this isomorphism is given by $\Gamma(\mathcal{G}(\Gamma(G, u)), [u]) \to \Gamma(G, u)$, $\langle [a], [0] \rangle \mapsto a$. By Theorem 7.2(i), we may extend this function to a homomorphism

$$h\colon \quad (\mathcal{G}(\Gamma(G, u)), [u]) \to G,$$
$$\langle [a_1, \ldots, a_m], [b_1, \ldots, b_n] \rangle \mapsto a_1 + \cdots + a_m - b_n - \cdots - b_1.$$

We note that words with entrances from $\Gamma(G, u)$ add up to the same element of G iff they are similar. Indeed, for groups in which (RDP_1) holds, we may prove a proposition analogous to Lemma 3.9. We conclude from this that two words of $\mathcal{W}(\Gamma(G, u))$ that sum up to the same element are similar. Clearly, similar words add up to the same element of G.

We claim that h is injective. Indeed, suppose we are given $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \in \mathcal{C}(\Gamma(G, u))$ such that $h(\langle \mathfrak{a}, \mathfrak{b} \rangle) = h(\langle \mathfrak{c}, \mathfrak{d} \rangle)$. Let $A \in G$ be the sum of the elements of any word representing $\mathfrak{a}$, and define $B, C, D \in G$ similarly; then the latter condition means $A - B = C - D$. It follows that $A + D_B = C + B$, where D_B is chosen such that $B + D_B = D + B$ holds. Now we have, according to Definition 5.5, $\mathfrak{b} + \mathfrak{d}_{\mathfrak{b}} = \mathfrak{d} + \mathfrak{b}$; hence any word representing $\mathfrak{d}_{\mathfrak{b}}$ adds up to D_B. So $\mathfrak{a} + \mathfrak{d}_{\mathfrak{b}} = \mathfrak{c} + \mathfrak{b}$, which means, by the same definition, $\langle \mathfrak{a}, \mathfrak{b} \rangle = \langle \mathfrak{c}, \mathfrak{d} \rangle$.

We claim that h is also surjective. Indeed, $\Gamma(G, u)$ generates G. For, because G fulfils (RIP), we have for $0 \le a \le b_1 + \cdots + b_n$ that $a = d_1 + \cdots + d_n$, where $0 \le d_1 \le b_1, \ldots, 0 \le d_n \le b_n$. We conclude that the unit interval of G generates the positive cone G^+. Moreover, since G possesses a strong unit, it is directed, which means, by Fuchs (1963), Proposition II.3(a), that G^+ generates G. $\square$

We may apply this result to the case where both the pseudoeffect algebras and the groups are lattice-ordered.

Definition 7.5. Let $\mathcal{LRPEA}$ denote the category whose objects are the lattice pseudoeffect algebras fulfilling (RDP_0) and whose morphisms are the homomorphisms of these structures.

Let $\mathcal{LG}$ denote the category whose objects are the unital ℓ-groups and whose morphisms are the homomorphisms of these structures.

Define Γ to be the functor from $\mathcal{LG}$ to $\mathcal{LRPEA}$ that maps an object (G, u) from $\mathcal{LG}$ to $\Gamma(G, u)$ according to Definition 2.1 and maps a morphism $\psi: (G, u) \to (H, v)$ from $\mathcal{LG}$ to its restriction to $\Gamma(G, u)$.

Define Δ to be the functor from $\mathcal{LRPEA}$ to $\mathcal{LG}$ that maps an object E from $\mathcal{LRPEA}$ to $(\mathcal{G}(E), [1])$ and maps a morphism $\varphi: E \to F$ from $\mathcal{LRPEA}$ to the morphism $\psi: \mathcal{G}(E) \to \mathcal{G}(F)$ subject to the conditions $\iota_F \circ \varphi = \psi \circ \iota_E$.

That Γ and Δ map indeed into $\mathcal{LRPEA}$ and $\mathcal{LG}$, respectively, follows from Propositions 3.3(ii) and 6.4(ii). That Γ and Δ are functors follows again by construction.

Theorem 7.6. *The functors $\Delta: \mathcal{LRPEA} \to \mathcal{LG}$ and $\Gamma: \mathcal{LG} \to \mathcal{LRPEA}$ form an equivalence of categories, that is, $\Gamma \circ \Delta$ and $\Delta \circ \Gamma$ are naturally equivalent to the identity functors of $\mathcal{LRPEA}$ and $\mathcal{LG}$, respectively.*

Proof: This follows from Theorem 7.4. □

8. RELATIONS BETWEEN PSEUDOEFFECT AND PSEUDO-MV ALGEBRAS

In the present section, we show how pseudo-MV algebras, which were introduced in Georgescu and Iorgulescu (to appear) are to be understood as a subfamily of the class of pseudoeffect algebras. With the help of our main Theorem 5.7, we are then able to present a new proof of the fact that pseudo-MV algebras are intervals in ℓ-groups (Dvurečenskij (to appear)).

Pseudo-MV algabras have been introduced in the following manner.

Definition 8.1. A structure $(M; \oplus, ^-, ^\sim, 0, 1)$, where $\oplus$ is a binary, $^-$ and $^\sim$ are unary operations, and $0, 1$ are constants, is called a *pseudo-MV algebra* if the following axioms hold in it.

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.
(A2) $x \oplus 0 = 0 \oplus x = x$.
(A3) $x \oplus 1 = 1 \oplus x = 1$.
(A4) $1^\sim = 0; 1^- = 0$.

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$.
(A6) $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^-) \oplus y = (y \odot x^-) \oplus x$.
(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$.
(A8) $x^{-\sim} = x$.

Here, for any $a, b, \in M$, we put $a \odot b \overset{\text{def}}{=} (b^- \oplus a^-)^\sim$.

Furthermore, we define, for $a, b \in M$

$$a \leq_M b \qquad \text{iff} \quad a \oplus c = b \quad \text{for some} \quad c \in M.$$

It can be shown that for any pseudo-MV algebra, $\leq_M$ is a lattice order (Georgescu and Iorgulescu (to appear), Proposition 1.10); the terms occurring in (A6) and (A7) are in fact the supremum and the infimum of two elements x and y, respectively.

By replacing the total operation $\oplus$ by a partial operation $+$, one can consider a pseudo-MV algebra as a pseudoeffect algebra.

Definition 8.2. Let (M; $\oplus, ^-, ^\sim$, 0, 1) be a pseudo-MV algebra. Define $+$ to be the partial operation on M that is defined for elements $a, b, \in M$ iff $a \leq_M b^-$, and in that case, let $a + b \overset{\text{def}}{=} a \oplus b$.

Theorem 8.3. *Let* $(M; \oplus, ^-, ^\sim, 0, 1)$ *be a pseudo-MV algebra. Then the following holds.*

(i) $(M; +, ^-, ^\sim, 0, 1)$ *is a pseudoeffect algebra.*
(ii) *Let $\leq$ be the order of M considered as a pseudoeffect algebra. Then this order coincides with the original one, i.e.,* $\leq_M = \leq$.
(iii) *For any* $a, b \in M$, *we have*

$$a \oplus b = (a \wedge b^-) + b. \tag{3}$$

Proof:

(ii) By Dvurečenskij and Pulmannová (2000), Remark 6.4.5, we have for any $a, b \in M$ that $a \leq_M b$ iff, for some $c \in M$, $a + c = b$. So $\leq_M = \leq$, where $\leq$ is defined according to Definition 1.5.
(iii) That Eq. (3) holds in a pseudo-MV algebra is seen, e.g., from Dvurečenskij and Pulmannová (2000), Proposition 6.4.8(i) and Exercise 3(1) of 6.4.5.
(i) Let $a, b, c \in M$. To show (E1), suppose $a + b$ and $(a + b) + c$ are defined. Then $b \leq a + b \leq c^-$, which means that $b + c$ is defined. Because we have $x \wedge y = (x \oplus y^\sim) \odot y$ for $x, y \in M$, from the assumptions $a + b \leq c^-$ and $a \leq b^-$ it follows that $a \wedge (b + c)^- = (a + b + c) \odot (b + c)^- = (a + b + c) \odot c^- \odot b^- = [(a + b) \wedge c^-] \odot b^- = (a + b) \odot b^- = a \wedge b^- = a$; so $a + (b + c)$ is defined. If $b + c$ and $a + (b + c)$

are defined, we conclude similarly that $a + b$ and $(a + b) + c$ are also defined. Now by (A1), (E1) follows.

Because we have $a + a^{\sim} = a^{-} + a = 1$ (Georgescu and Iorgulescu (to appear), Proposition 1.5) and because cancellation holds for the operation (Dvurečenskij and Pulmannová (2000), Proposition 6.4.4), (E2) and (EC) follow.

To show (E3), let $a + b$ be defined. Then $a, b \le a + b$, and we have $a + b = c + a = b + d$ for exactly one element c and one element d (Dvurečenskij and Pulmannová (2000), Remark 6.4.5 and Proposition 6.4.4), so (E3) follows.

If $1 + a$ or $a + 1$ is defined, $a \le 0$, i.e., $a = 0$. So (E4) follows. □

We are now able to give a new, completely different proof of the main result of Dvurečenskij (to appear).

Theorem 8.4. *Let $(M; \oplus, ^{-}, ^{\sim}, 0, 1)$ be a pseudo-MV algebra. Then there is an ℓ-group $(G; +, \le)$ with strong unit u such that $(M; \oplus, ^{-}, ^{\sim}, 0, 1)$ and $(\Gamma(G, u); \oplus, ^{-}, ^{\sim}, 0, u)$ are isomorphic, where for $a, b \in \Gamma(G, u)$, we define*

$$a \oplus b = (a + b) \wedge u, \tag{4}$$

$$a^{-} = u - a, \tag{5}$$

$$a^{\sim} = -a + u. \tag{6}$$

Proof: We know from Theorem 8.3 that $(M; +, ^{-}, ^{\sim}, 0, 1)$ is a pseudoeffect algebra whose order $\le$ coincides with $\le_M$. From this and from Dvurečenskij and Pulmannová (2000), Theorem 6.4.12, it follows that M as a pseudoeffect algebra fulfils (RDP_2).

So by Theorem 5.7 and Proposition 6.4 there is a unital ℓ-group (G, u) such that $(M; +, ^{-}, ^{\sim}, 0, 1)$ is isomorphic to $(\Gamma(G, u); +, ^{-}, ^{\sim}, 0, u)$.

Here, according to the remark following Definition 2.1, $^{-}, ^{\sim}\colon \Gamma(G, u) \to \Gamma(G, u)$ are given by (5) and (6), respectively.

It remains to check that under this isomorphism, $\oplus\colon \Gamma(G, u) \to \Gamma(G, u)$ as defined by (4) coincides with the equally denoted operation of the pseudo-MV algebra M. Now, for any $a, b \in \Gamma(G, u)$, we have $a \oplus b = (a + b) \wedge u = (a + b) \wedge (b^{-} + b) = (a \wedge b^{-}) + b$, where $\wedge$ and $+$ are calculated in the group G. Now the last term may be equally calculated in the pseudoeffect algebra $(\Gamma(G, u); +, ^{-}, ^{\sim}, 0, u)$. So by Theorem 8.3(iii), the claim follows. □

Now, under certain presumptions, a pseudoeffect algebra may also be considered as a pseudo-MV algebra.

Definition 8.5. Let $(E; +, 0, 1)$ be a pseudoeffect algebra. Let $/$ and $\backslash$ be partial binary operations such that, for $a, b \in E$, a/b and $a\backslash b$ are defined iff $b \le a$, in which case, the conditions

$$a = a/b + b = b + a\backslash b$$

are to be fulfilled.

Furthermore, let E be lattice-ordered. Let $\oslash$ and $\circledbslash$ be the binary operations defined for $a, b \in E$ by

$$a \oslash b \stackrel{\text{def}}{=} a/(a \wedge b),$$
$$a \circledbslash b \stackrel{\text{def}}{=} a\backslash(a \wedge b).$$

Finally, let, for $a, b \in E$,

$$a \oplus b \stackrel{\text{def}}{=} (b^- \oslash a)^\sim.$$

Lemma 8.6. *Let* $(E; +, 0, 1)$ *be a pseudoeffect algebra.*

(i) *If, for* $a, b \in E$, $b \le a$, *then we have*

$$a\backslash b = b^\sim/a^\sim, \qquad (7)$$
$$a/b = b^-\backslash a^-.$$

(ii) *Let* E *be lattice-ordered. If* $a + b$ *is defined, we have* $a \oplus b = a + b$.

Proof:

(i) Let $b \le a$. Then $a\backslash b = x$ means $a = b + x$, so by Lemma 1.4(vi), $b^\sim = x + a^\sim$ and $x = b^\sim/a^\sim$. So the first equation of (7) follows, from which the second is easily derivable.
(ii) Let $a + b$ exist. So by Lemma 1.6(iv), $a \le b^-$ holds. Then we have $b^- \oslash a = b^-/(a \wedge b^-) = b^-/a$. From $b^-/a = c^-$ it follows that $b^- = c^- + a$, and so $c = a + b$ by Lemma 1.4(vi). This means that $a \oplus b = (b^- \oslash a)^\sim = (b^-/a)^\sim = c = a + b$. □

The following lemma contains the conditions under which a pseudoeffect algebra may be considered a pseudo-MV algebra.

Proposition 8.15. *Let* $(E; +, 0, 1)$ *be a lattice pseudoeffect algebra. Then the following statements are equivalent.*

(α) *E fulfils* (RDP_0).
(β) *E fulfils* (RDP_2).

(γ) *For all* $a, b \in E$,

$$a/(a \wedge b) = (a \vee b)/b. \tag{8}$$

(δ) *For all* $a, b \in E$

$$a\backslash(a \wedge b) = (a \vee b)\backslash b. \tag{9}$$

If one of these statements is true, we have, for $a, b \in E$

$$a \oslash b = b^{\sim} \oslash a^{\sim}, \tag{10}$$
$$a \oslash b = b^{-} \oslash a^{-}.$$

Proof: (α) $\Leftrightarrow$ (β). This has been proved in Proposition 3.3(ii).

(β) $\Rightarrow$ (γ). Let E fulfil (RDP_2). So from $a^- + a = b^- + b$, we get elements $d_1, \ldots, d_4 \in E$ such that

$$\begin{array}{cccc} d_1 & d_2 & \rightarrow & b^- \\ d_3 & d_4 & \rightarrow & b \\ \downarrow & \downarrow & & \\ a^- & a & & \end{array}$$

and $d_2 \wedge d_3 = 0$. By Lemma 1.7(i), we have $d_1 = d_1 + (d_2 \wedge d_3) = a^- \wedge b^-$ and similarly $d_4 = a \wedge b$. So by (7), we have $a/(a \wedge b) = a/d_4 = d_2 = b^-\backslash d_1 = b^-\backslash(a^- \wedge b^-) = b^-\backslash(a \vee b)^- = (a \vee b)/b$. So (8) is proved.

(γ) $\Rightarrow$ (δ). Suppose (8). Using (7), it follows for any $a, b \in E$ that $a\backslash(a \wedge b) = (a \wedge b)^{\sim}/a^{\sim} = (a^{\sim} \vee b^{\sim})/a^{\sim} = b^{\sim}/(a^{\sim} \wedge b^{\sim}) = b^{\sim}/(a \vee b)^{\sim} = (a \vee b)\backslash b$. So (9) holds.

(δ) $\Rightarrow$ (α). Suppose (9), and let $a \leq b + c$. Then $a = (a \wedge b) + (a\backslash(a \wedge b))$; we have $a \wedge b \leq b$ and, because $b + ((a \vee b)\backslash b) = a \vee b \leq b + c$, also $a\backslash(a \wedge b) = (a \vee b)\backslash b \leq c$. So ($\mathrm{RDP}_0$) is shown.

Now suppose that (δ) holds, and let $a, b \in E$. By (7), we have $a \oslash b = a\backslash(a \wedge b) = (a \vee b)\backslash b = b^{\sim}/(a^{\sim} \wedge b^{\sim}) = b^{\sim} \oslash a^{\sim}$. This is the first equation of (10), from which the second is easily derivable. □

Theorem 8.7. *Let* $(E; +, ^{\sim}, ^{-}, 0, 1)$ *be a lattice pseudoeffect algebra. Then* $(E; \oplus, ^{\sim}, ^{-}, 0, 1)$ *is a pseudo-MV algebra if and only if* E *fulfils one of the equivalent conditions of Proposition 8.7. In that case, we have for* $a, b \in E$

$$a \oplus b = (b^{-} \oslash a)^{\sim} = (a^{\sim} \oslash b)^{-}. \tag{11}$$

$$a \odot b = a \oslash b^{\sim} = b \oslash a^{-}. \tag{12}$$

Moreover, the order $\leq_M$ of E as an MV-algebra then coincides with the order $\leq$ of E as a pseudoeffect algebra, and we have for $a, b \in E$

$$a \wedge b = a \oslash (a \obslash b) = a \obslash (a \oslash b), \tag{13}$$

$$a \vee b = (a \oslash b) \oplus b = a \oplus (b \obslash a). \tag{14}$$

Proof: Let $(E; \oplus, ^{\sim}, ^{-}, 0, 1)$ be a pseudo-MV algebra.

We show first that the order $\leq_M$ of E coincides with the original one, that is, with $\leq$. Let $a, b \in E$. From a $\leq_M b$, it follows that $b = c \oplus a$ for some $c \in E$; so $b^- = a^- \oslash c$, that is, $a^- = b^- + (a^- \wedge c)$; but this means $b^- \leq a^-$ and, by Lemma 1.6(iii), $a \leq b$. Conversely, $a \leq b$ means $b = a + d$ for some $d \in E$; so $b = a \oplus d$ and hence $a \leq_M b$.

It follows for $a, b \in E$ that $a + b$ exists iff a $\leq_M b^-$, in which case, $a + b = a \oplus b$. This means that the $+$ operation is derivable from the pseudo-MV algebra M as described in Definition 8.2. So we may conclude from Dvurečenskij and Pulmannová (2000), Theorem 6.4.11, that E as a pseudoeffect algebra fulfills (RDP_0), that is, that condition (α) of Proposition 8.7 is fulfilled.

Let us now assume that conditions (γ) and (δ) of Proposition 8.7 are fulfilled, that is, that (8) and (9) hold.

We see equally as above that the order $\leq$ of E coincides with the relation $\leq_M$ as defined in Definition 8.1 for pseudo-MV algebras.

We will now prove Eq. (11)–(14) and then every one of the axioms (A1)–(A8) of a pseudo-MV algebra.

Let $a, b, x \in E$. We have $a \oplus b = (b^- \oslash a)^{\sim} = x$ iff $b^- \oslash a = b^-/(a \wedge b^-) = x^-$ iff $b^- = x^- + a \wedge b^-$ iff $x = (a \wedge b^-) + b$ iff $(a \wedge b^-)^{\sim} = b + x^{\sim}$ iff $(a^{\sim} \vee b)\backslash b = x^{\sim}$ iff $a^{\sim}\backslash(a^{\sim} \wedge b) = x^{\sim}$ iff $(a^{\sim} \obslash b)^- = x$. That is, (11) holds.

Now, by (11) and (10), we get $a \odot b = (b^- \oplus a^-)^{\sim} = b \obslash a^- = a \oslash b^{\sim}$. That is, (12) holds.

From $(a \wedge b) + (a \obslash b) = a$, we conclude $a \wedge b = a/(a \obslash b) = a \oslash (a \obslash b)$, and similarly we get $a \wedge b = a\backslash(a \oslash b) = a \obslash (a \oslash b)$. So (13) is shown.

Using this result, we conclude by (11) that $a \vee b = (a^- \wedge b^-)^{\sim} = [b^- \oslash (b^- \obslash a^-)]^{\sim} = [b^- \oslash (a \oslash b)]^{\sim} = (a \oslash b) \oplus b$, and similarly we get $a \vee b = (a^{\sim} \wedge b^{\sim})^- = [a^{\sim} \obslash (a^{\sim} \oslash b^{\sim})]^- = [a^{\sim} \obslash (b \obslash a)]^- = a \oplus (b \obslash a)$. So (14) is shown. Now, (A4) and (A8) hold in E by (iii) and (iv) of Lemma 1.4.

For $a \in E$, from $(1 \oslash a) + a = 1$, we have $1 \oslash a = a^-$, so $a \oplus 0 = (0^- \oslash a)^{\sim} = (1 \oslash a)^{\sim} = a^{-\sim} = a$, which proves the first part of (A2). The second is shown analogously.

We have $a \oplus 1 = (1^- \oslash a)^{\sim} = (0 \oslash a)^{\sim} = (0/0)^{\sim} = 0^{\sim} = 1$, which proves the first part of (A3). The second is shown analogously.

For $a, b \in E$, we have by (11) and (12) that $(a^{\sim} \oplus b^{\sim})^- = b \oslash a^{\sim} = a \obslash b^- = (a^- \oplus b^-)^{\sim}$, which proves (A5).

All the expressions occurring in (A6) equal the supremum of a and b. For, by (14), we have $a \oplus (a^{\sim} \odot b) = a \oplus (b \oslash a) = a \vee b$, hence also $b \oplus (b^{\sim} \odot a) = a \vee b$. Similarly, $(a \odot b^{-}) \oplus b = (a \oslash b) \oplus b = a \vee b$, hence also $(b \odot a^{-}) \oplus a = a \vee b$. The expressions occurring in (A7) equal the infimum of a and b; for by (13), we have $a \odot (a^{-} \oplus b) = a \oslash (a \oslash b) = a \wedge b$ and $(a \oplus b^{\sim}) \odot b = b \oslash (b \oslash a) = a \wedge b$.

To prove associativity, that is, (A1), several intermediate results are needed, denoted by $(a0), \ldots, (a5)$. Let $a, b, c, x, y \in E$.

Claim (a0). From $a \leq x \leq y$, it follows that $x \backslash a \leq y \backslash a$.

For: For some x', we have $x + x' = y$, and so $a + (x \backslash a) + x' = a + (y \backslash a)$ and $(x \backslash a) + x' = y \backslash a$.

Claim (a1). The order is monotone from both sides with respect to $\oplus$ and $\odot$.

For: From $a \leq b$, it follows that $b^{\sim} \vee x \leq a^{\sim} \vee x$ and, by (a0), $b^{\sim} \oslash x = (b^{\sim} \vee x) \backslash x \leq (a^{\sim} \vee x) \backslash x = a^{\sim} \oslash x$; so $a \oplus x = (a^{\sim} \oslash x)^{-} \leq (b^{\sim} \oslash x)^{-} = b \oplus x$. Similarly, we conclude $x \oplus a \leq x \oplus b$.

Moreover, from $a \leq b$ it follows that $b^{-} \leq a^{-}$, so $b^{-} \oplus x^{-} \leq a^{-} \oplus x^{-}$; so $x \odot a = (a^{-} \oplus x^{-})^{\sim} \leq (b^{-} \oplus x^{-})^{\sim} = x \odot b$. Similarly, we conclude that $a \odot x \leq b \odot x$.

Claim (a2). $a \odot b \leq c$ iff $b \leq a^{-} \oplus c$ iff $a \leq c \oplus b^{\sim}$.

For: Suppose $a \odot b \leq c$. Then for some $x \in E$, we have $(a \odot b) + x = b \oslash a^{-} + x = c$. This means that $b + x = (a^{-} \wedge b) + c$. So $b \leq (a^{-} \wedge b) + c = (a^{-} \wedge b) \oplus c \leq a^{-} \oplus c$ by (a1).

Suppose $b \leq a^{-} \oplus c$. Then again by (a1), we have $a \odot b \leq a \odot (a^{-} \oplus c) = a \wedge c \leq c$.

Similarly, we prove the equivalence of $a \odot b \leq c$ with $a \leq c \oplus b^{\sim}$.

Claim (a3). $a \oplus (b \wedge c) = (a \oplus b) \wedge (a \oplus c)$; $(b \vee c) \odot a = (b \odot a) \vee (c \odot a)$.

For: $a \oplus (b \wedge c) \leq a \oplus b, a \oplus c$ by (a1). Suppose $x \leq a \oplus b, a \oplus c$. Then by (a2), $a^{\sim} \odot x \leq b, c$, so $a^{\sim} \odot x \leq b \wedge c$, and again by (a2), $x \leq a \oplus (b \wedge c)$. So the first equation follows. The second follows from the first one, applied to a^{-}, b^{-}, and c^{-}.

Analogously we may prove the following:

Claim (a3′). $(b \wedge c) \oplus a = (b \oplus a) \wedge (c \oplus a)$; $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$.

Claim (a4). The order is distributive.

For: By (a3) and (a1), we have $a \wedge (b \vee c) = (b \vee c) \odot [(b \vee c)^{-} \oplus a] = (b \odot [(b \vee c)^{-} \oplus a]) \vee (c \odot [(b \vee c)^{-} \oplus a]) \leq [b \odot (b^{-} \oplus a)] \vee [c \odot (c^{-} \oplus a)] = (a \wedge b) \vee (a \wedge c)$.

Claim (a5). $a \oplus (b \vee c) = (a \oplus b) \vee (a \oplus c)$.

For: By (a1), we have $a \oplus b, a \oplus c \leq a \oplus (b \vee c)$. Suppose $a \oplus b, a \oplus c \leq x$. Then by (a1), $a = a \oplus 0 \leq a \oplus b \leq x$ and $a^\sim \wedge b = a^\sim \odot (a \oplus b) \leq a^\sim \odot x$; similarly, $a^\sim \wedge c \leq a^\sim \odot x$. So by (a3), (a4), and (a1), we get $a \oplus (b \vee c) = (a \oplus a^\sim) \wedge [a \oplus (b \vee c)] = a \oplus [a^\sim \wedge (b \vee c)] = a \oplus [(a^\sim \wedge b) \vee (a^\sim \wedge c)] \leq a \oplus (a^\sim \odot x) = a \vee x = x$.

Analogously, using (a3′) instead of (a3), we may prove the following:

Claim (a5′). $(b \vee c) \oplus a = (b \oplus a) \vee (c \oplus a)$.

We finally prove the associativity of $\odot$; (A1) then easily follows. That is, we claim $(a \odot b) \odot c = a \odot (b \odot c)$.

First, set $d = (a \odot b)^- \vee c$. Then, by (a3′), $(a \odot b) \odot d = (a \odot b) \odot [(a \odot b)^- \vee c] = (a \odot b) \odot c$, and $a \odot (b \odot d) = a \odot [b \odot ((a \odot b)^- \vee c)] = a \odot [(b \odot (b^- \oplus a^-)) \vee (b \odot c)] = a \odot [(a^- \wedge b) \vee (b \odot c)] = [a \odot (a^- \wedge b)] \vee [a \odot (b \odot c)] = a \odot (b \odot c)$.

So we must show $(a \odot b) \odot d = a \odot (b \odot d)$. We have $a^- \leq d$, because $(a \odot b) \wedge c^\sim \leq a \odot b \leq a \odot 1 = a$, i.e., $a^- \leq [(a \odot b) \wedge c^\sim]^- = d$. So $a^- + d^\sim$ exists.

By (a5′), we have $b \vee (a^- + d^\sim) = b \vee d^\sim \vee (a^- + d^\sim) = [(b \vee d^\sim)/d^\sim + d^\sim] \vee (a^- + d^\sim) = [(b \oslash d^\sim) \vee a^-] + d^\sim$ and by (a5), $b \vee (a^- + d^\sim) = a^- \vee b \vee (a^- + d^\sim) = [a^- + ((a^- \vee b)\backslash a^-)] \vee (a^- + d^\sim) = a^- + [(b \obslash a^-) \vee d^\sim]$, so that $[(b \oslash d^\sim) \vee a^-] + d^\sim = a^- + [(b \obslash a^-) \vee d^\sim]$. Because of the associativity holding in E, we may rewrite this equation in the form $a^- + [((b \oslash d^\sim) \vee a^-)\backslash a^-] + d^\sim = a^- + [((b \obslash a^-) \vee d^\sim)/d^\sim] + d^\sim$ and conclude that $((b \oslash d^\sim) \vee a^-)\backslash a^- = ((b \obslash a^-) \vee d^\sim)/d^\sim$. But this means $(b \oslash d^\sim) \obslash a^- = (b \obslash a^-) \oslash d^\sim$, or $(a \odot b) \odot d = a \odot (b \odot d)$. □

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